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Autoresonance - A Kinetic Perspective

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Outline

- Introduction: the Autoresonant pendulum.
- Thermal broadening of the threshold.
- Quantum saturation at low temperatures.
- The self field effect.
- The quantum counterpart: Ladder climbing.
- Summary

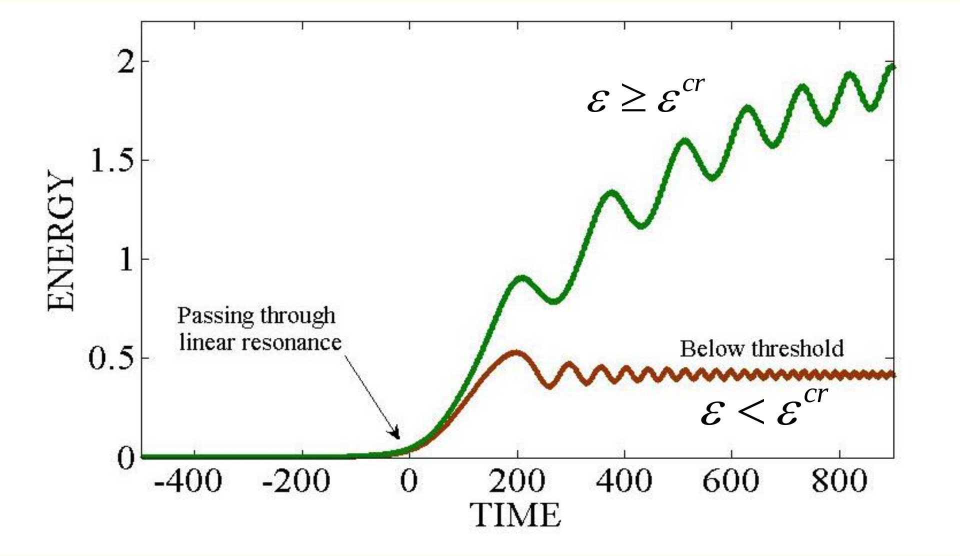
Collaborators:

Lazar Friedland (HUJI), Arkadiy Shagalov (Ekaterinburg), Eli Sarid (CERN)
Kater Murch & Irfan Siddiqi (Berkeley), Yoni Shalibo & Nadav Katz (HUJI)

Supported by the Israel Science Foundation (Grants: 1080/06, 451/10).

Introduction: The Autoresonant Pendulum

- A Chirped driven pendulum: $\ddot{x}+\sin x=\varepsilon\cos\psi_d$
- Chirped driving frequency: $\omega_d(t)=\dot{\psi}_d=1-\alpha t$
- Passage through linear resonance at t = 0.
- A continuing *phase-locking* is established and preserved.
- Threshold for capture into AR: $\varepsilon\geq\varepsilon^{cr}\propto\alpha^{3/4}$



Movie : http://socrates.berkeley.edu/~fajans/Autoresonance

Previous & Current Work

- Nonlinear oscillators
- Atomic and molecular physics
- Fluids and plasmas (e.g. BGK modes)
- Planetary dynamics
- Nonlinear optics
- Josephson junctions
- Antihydrogen project (CERN)

Recent Application: Antihydrogen Project (CERN)

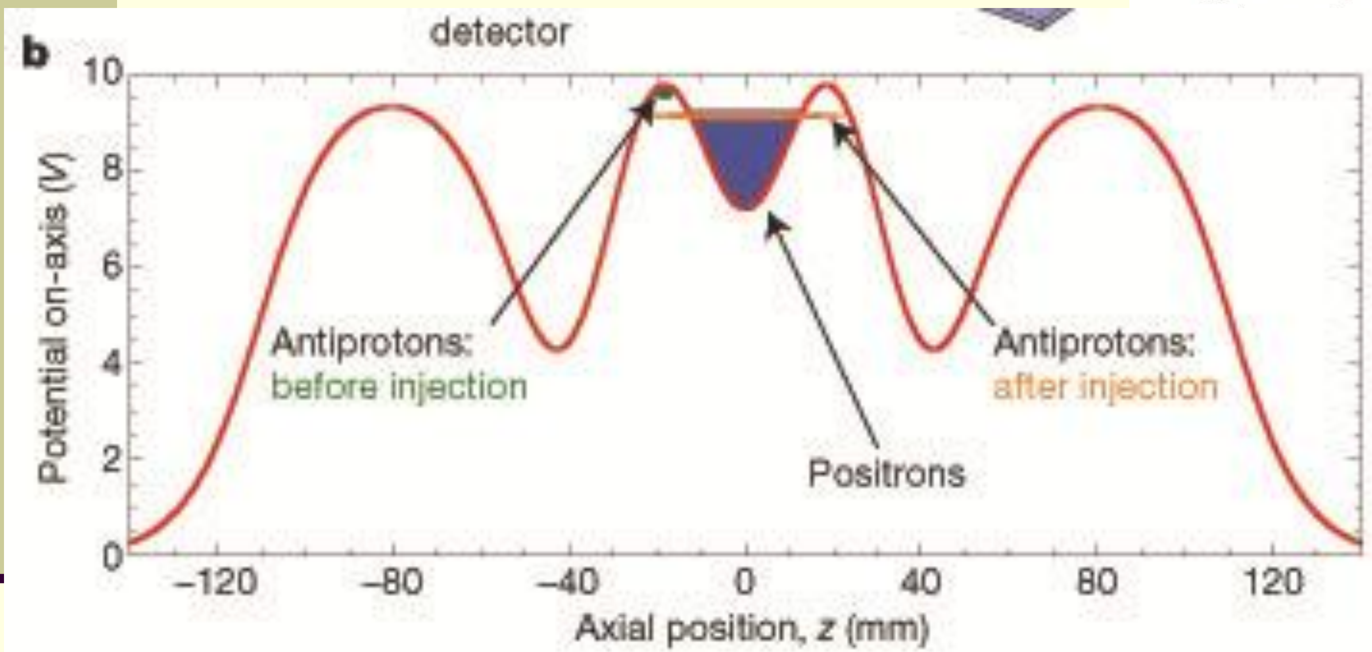
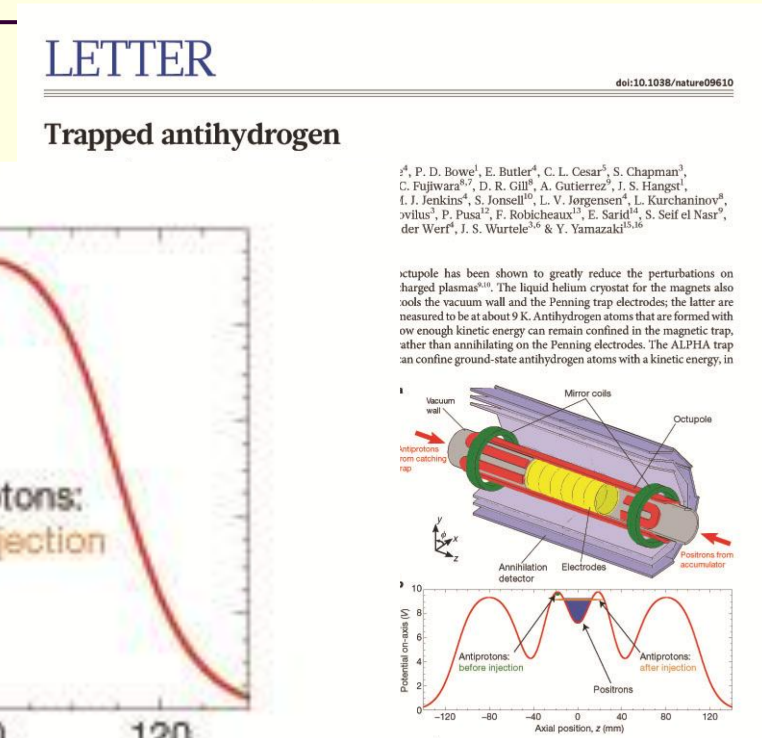


Figure 1 | The ALPHA central apparatus and mixing potential.
a, Antihydrogen synthesis and trapping region of the ALPHA apparatus. The atom-trap magnets, the modular annihilation detector and some of the Penning trap electrodes are shown. An external solenoid (not shown) provides


LETTER

doi:10.1038/nature09610

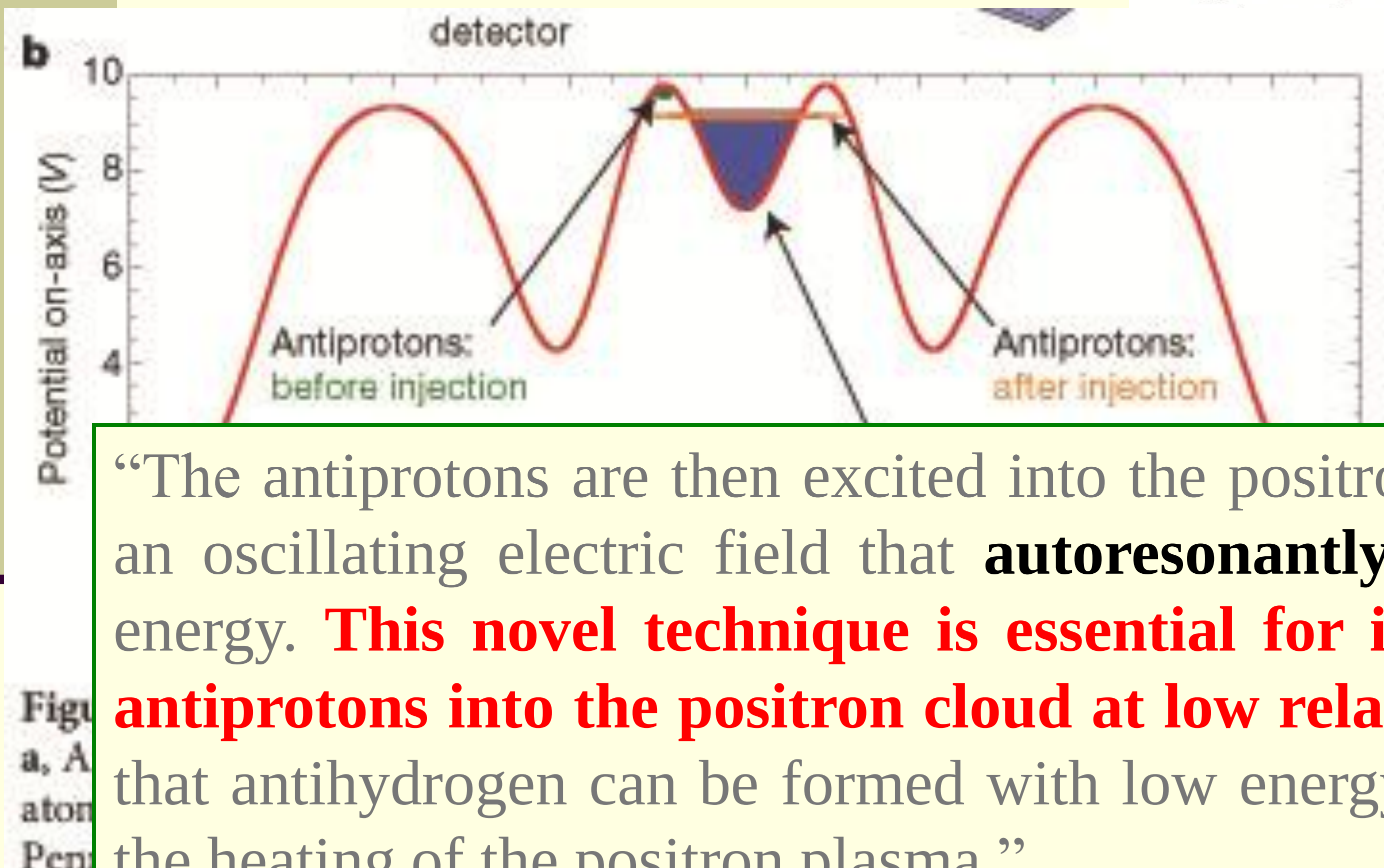
Trapped antihydrogen

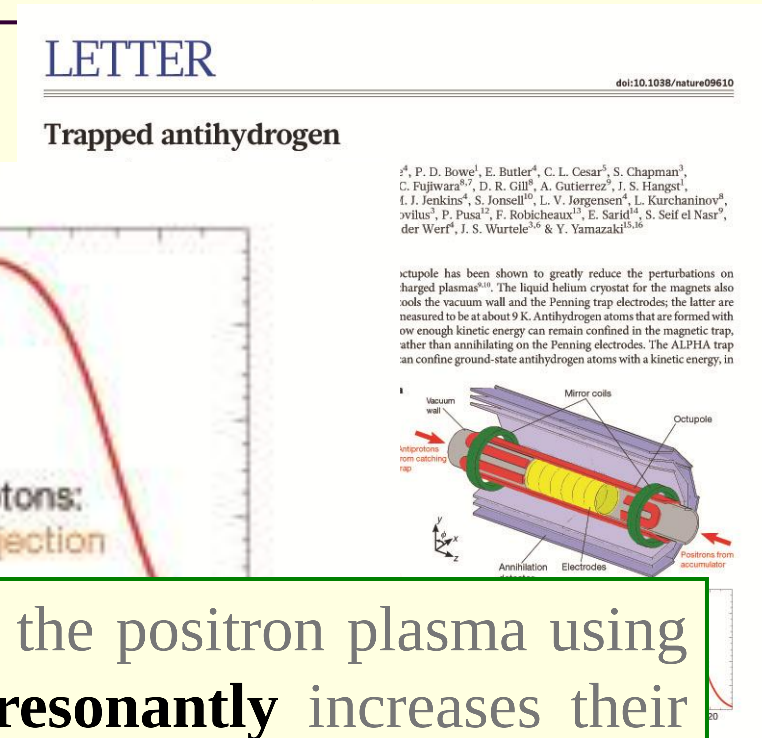
...[4], P. D. Bowe[1], E. Butler[4], C. L. Cesar[5], S. Chapman[3], ...C. Fujiwara[8,7], D. R. Gill[8], A. Gutierrez[9], J. S. Hangst[1], ...J. Jenkins[4], S. Jonsell[10], L. V. Jørgensen[4], L. Kurchaninov[8], ...ovilus[3], P. Pusa[12], F. Robicheaux[13], E. Sarid[14], S. Seif el Nasr[9], ...der Werf[4], J. S. Wurtele[3,6] & Y. Yamazaki[15,16]

...ctupole has been shown to greatly reduce the perturbations on ...harged plasmas[9,10]. The liquid helium cryostat for the magnets also ...cools the vacuum wall and the Penning trap electrodes; the latter are ...measured to be at about 9 K. Antihydrogen atoms that are formed with ...ow enough kinetic energy can remain confined in the magnetic trap, ...ather than annihilating on the Penning electrodes. The ALPHA trap ...can confine ground-state antihydrogen atoms with a kinetic energy, in

Andresen et. al., Nature **468** (2010).

Recent Application: Antihydrogen Project (CERN)




LETTER

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Trapped antihydrogen

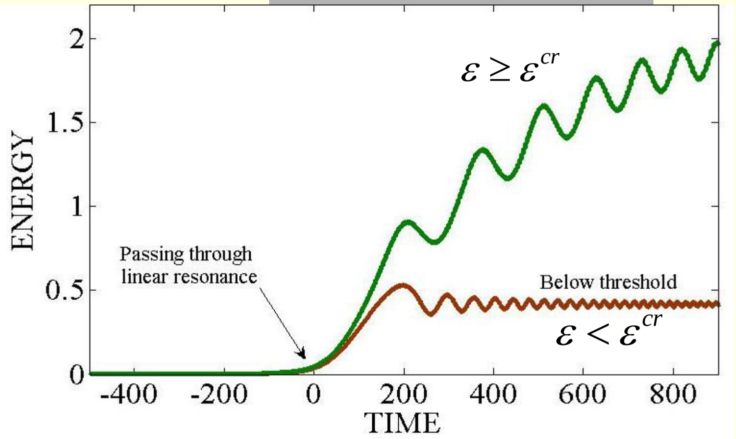
"The antiprotons are then excited into the positron plasma using an oscillating electric field that **autoresonantly** increases their energy. **This novel technique is essential for introducing the antiprotons into the positron cloud at low relative velocity,** so that antihydrogen can be formed with low energy, and to reduce the heating of the positron plasma."

Andresen et. al., Nature **468** (2010).

A. Thermal broadening of the threshold

Chirped driven pendulum: $\ddot{x} + \sin x = \varepsilon \cos \psi_d$

$$\omega_d(t) = \dot{\psi}_d = 1 - \alpha t$$



Sharp Threshold: $\varepsilon \geq \varepsilon^{cr} \propto \alpha^{3/4}$

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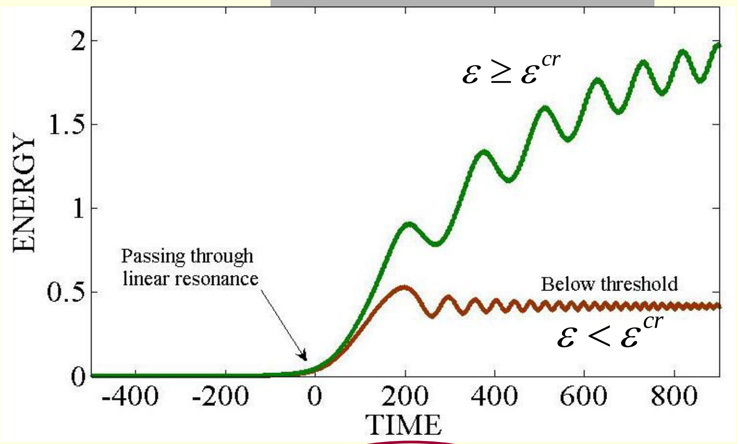
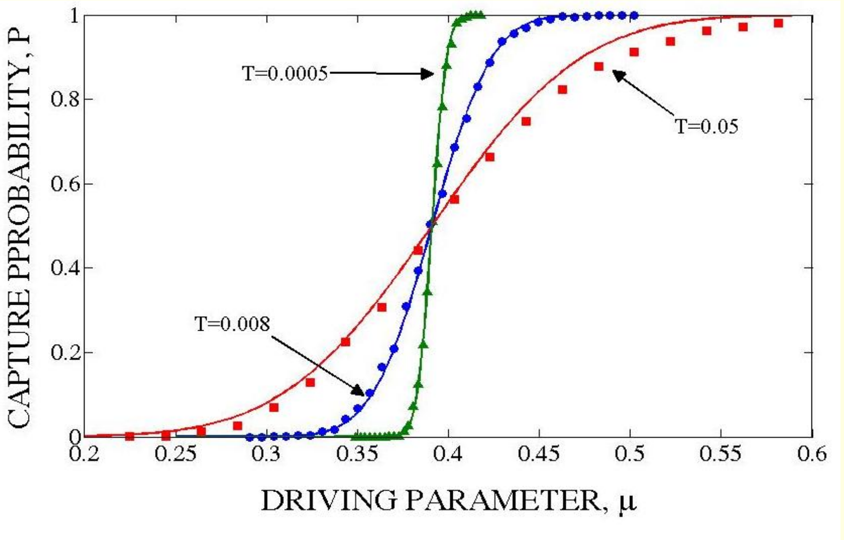


Sharp Threshold: $\varepsilon \geq \varepsilon^{cr} \propto \alpha^{3/4}$

If we add thermal noise we widen the transition: $\ddot{x}+\nu\dot{x}+\sin x=\varepsilon\cos\psi_d+\sqrt{2\nu T}\xi(t)$

Transition width, $\Delta\varepsilon \propto \sqrt{\alpha T}$



Similar effect is observed for thermal distributed non-interacting particles.

Barth, Friedland, Sarid, and Shagalov, PRL **103,** 155001 (2009)

Thermal broadening - Theory

- **Assumptions:**
  - Sufficiently small temperature → Weak nonlinearity.
  - Fast passage through resonance → Neglect noise.
  - Noise & dissipation form thermal initial conditions, f_T (FDT theorem).
- **The idea:** find *P(initial conditions)* and integrate over f_T.

Thermal broadening - Theory

- **Assumptions:**
  - Sufficiently small temperature → Weak nonlinearity.
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- **The idea:** find *P(initial conditions)* and integrate over f_T.
- **Analysis: single resonance approximation**
  - Seek solution of form: $x \propto A\cos\theta$.
  - Neglect non resonant terms.
  - Average over the fast phase.
  - Complex variables: $\Psi \propto Ae^{i\phi}$
  - Phase mismatch: $\phi = \theta - \psi_d$
  - Dimensionless time: $\tau = \alpha^{1/2} t$
  - Damping: $\gamma \propto \nu\alpha^{-1/2}$

$$\ddot{x} + \nu\dot{x} + \sin x = \varepsilon\cos\psi_d$$

$$\psi_d = t - \alpha t^2/2$$

$$i\frac{d\Psi}{d\tau} + \left(|\Psi|^2 - \tau\right)\Psi + i\gamma\Psi = \mu$$

Driving Parameter: $\mu \propto \varepsilon\alpha^{-3/4}$

Thermal broadening - Theory

- Weakly nonlinear equation (NLS-type) $i\dot{\Psi} + \left(|\Psi|^2 - \tau\right)\Psi + i\gamma\Psi = \mu$
- Phase locked asymptotic solution: $|\Psi|^2 \approx \tau, \quad \phi = const$
- Critical driving parameter: $\mu > \mu_{cr}(\gamma, A_0, \phi_0) \approx c_0(\gamma) + \kappa A_0 \cos\phi_0$

$\kappa = 0.245, \quad c_0 \approx 0.4 \quad A_0 \leq 0.3$

- Resonant capture probability: $P(\mu, A_0) = \begin{cases} 0 & \mu < c_0 - \kappa A_0 \\ \frac{1}{\pi}\arccos\left(\frac{c_0 - \mu}{\kappa A_0}\right) & |\mu - c_0| < \kappa A_0 \\ 1 & \mu > c_0 + \kappa A_0 \end{cases}$

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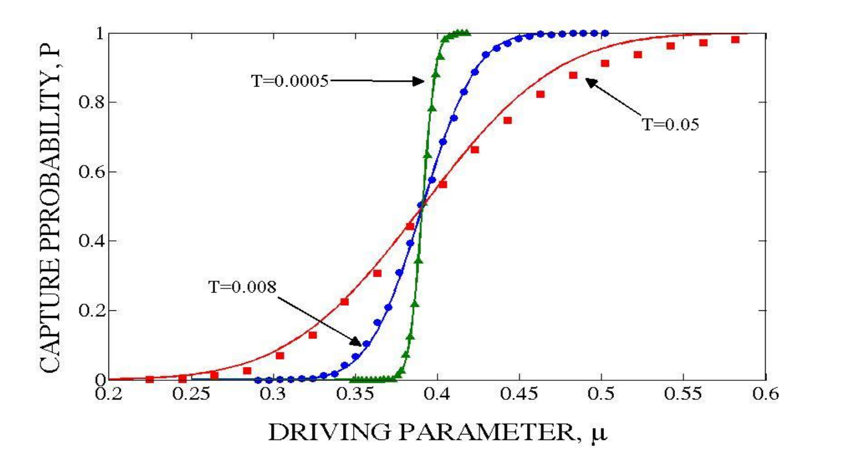
- Integration:

$$P\left(\mu\right) = \int_0^\infty P\left(\mu, A_0\right) f_T\left(A_0\right) dA_0$$

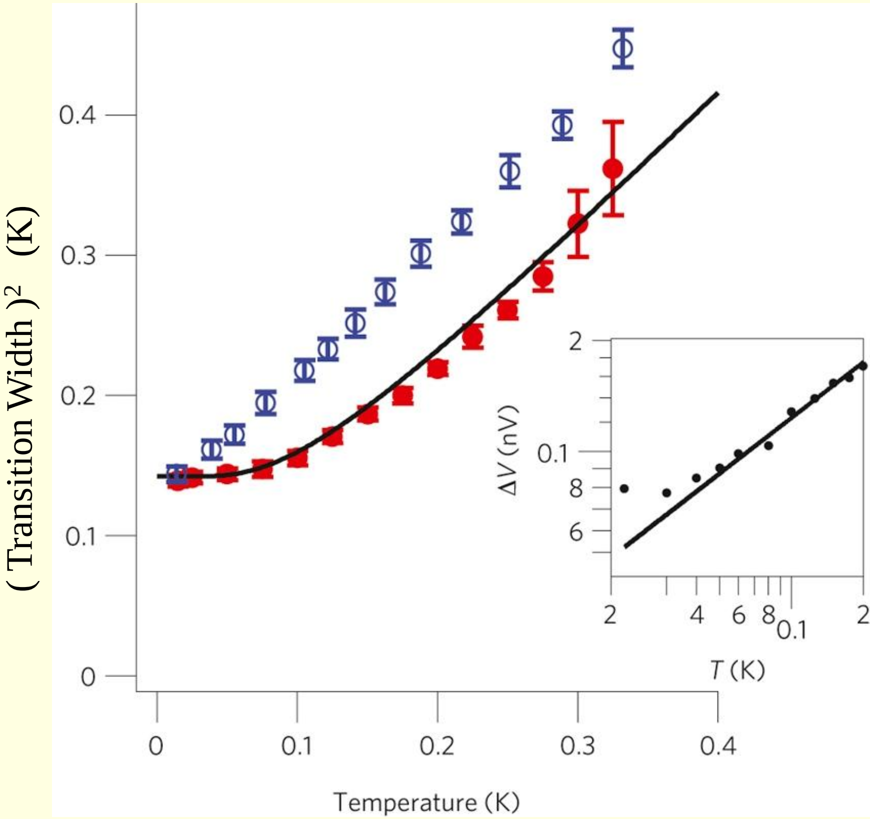
Transition width $\propto \sqrt{T}$



Barth, Friedland, Sarid, and Shagalov, PRL **103,** 155001 (2009)

B. Quantum saturation

- The transition width at low temperatures:



Josephson junctions experiment

Murch, Vijay, Barth, Naaman, Aumentado, Friedland, and Siddiqi, Nature Physics **7,** 105 (2011).

B. Quantum saturation

- The transition width at low temperatures:

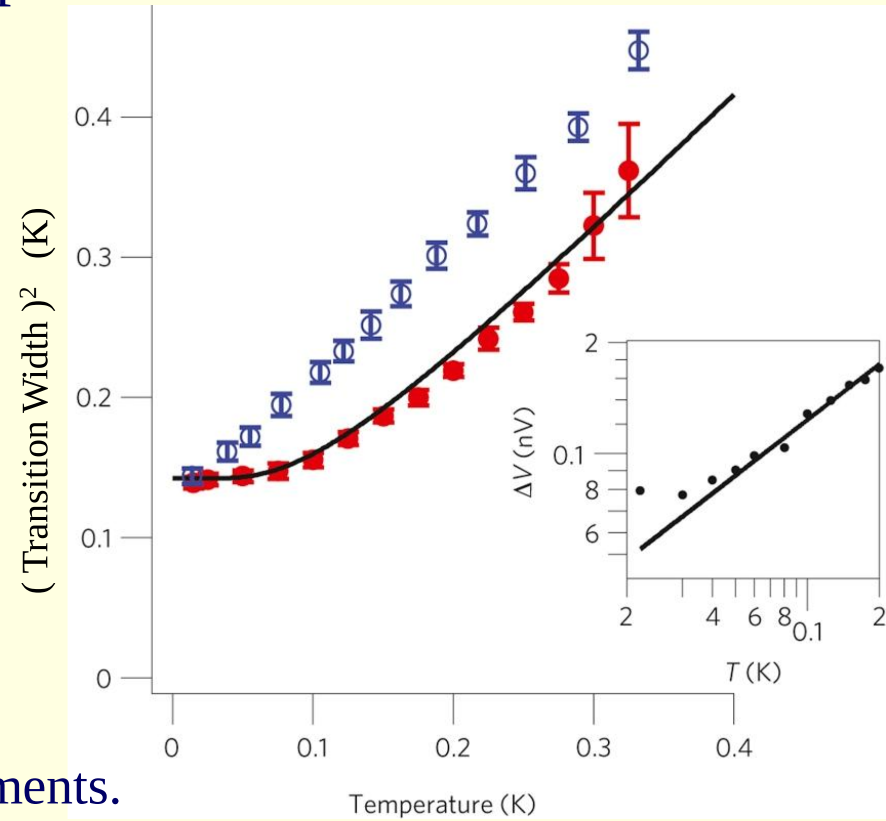
$$\text{Transition width } \Delta\varepsilon \propto \sqrt{T} \rightarrow \sqrt{T_{eff}}$$

- Effective temperature:

$$T_{eff} = \frac{\hbar\omega_0}{2k_B}\coth\left(\frac{\hbar\omega_0}{2k_B T}\right) \begin{cases} \xrightarrow{\hbar\omega_0 << k_B T} k_B T \\ \xrightarrow{k_B T << \hbar\omega_0} \frac{\hbar\omega_0}{2} \end{cases}$$

- Theory: Wigner function

- Applications:
    - Resolution limit for qbit measurements.
    - Quantum noise thermometer



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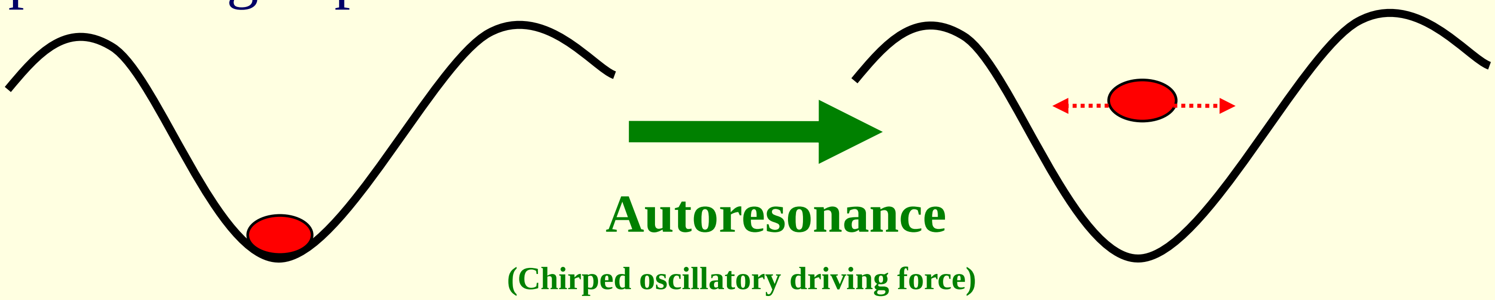
C. The self-field effect

- Trapped charged particle cloud



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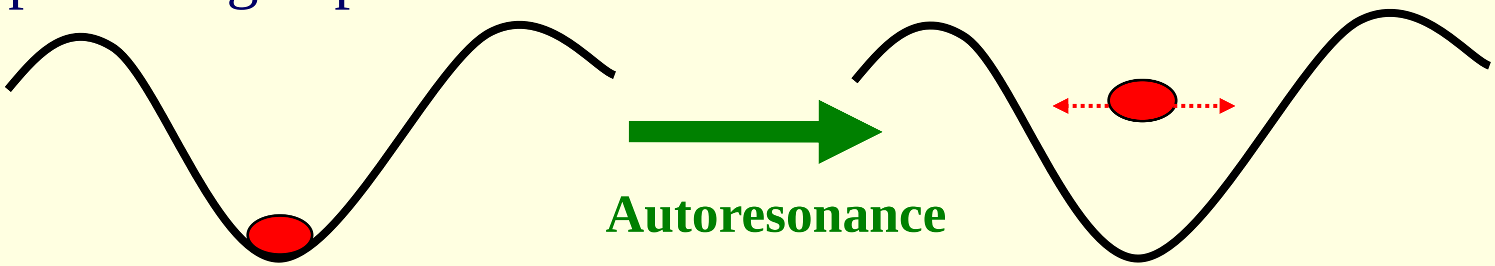
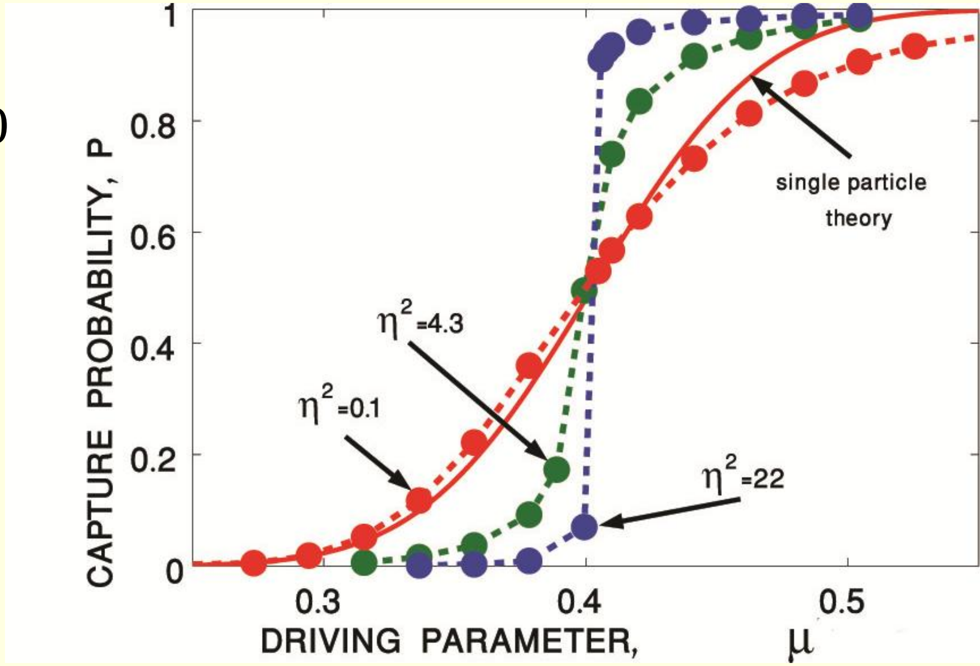


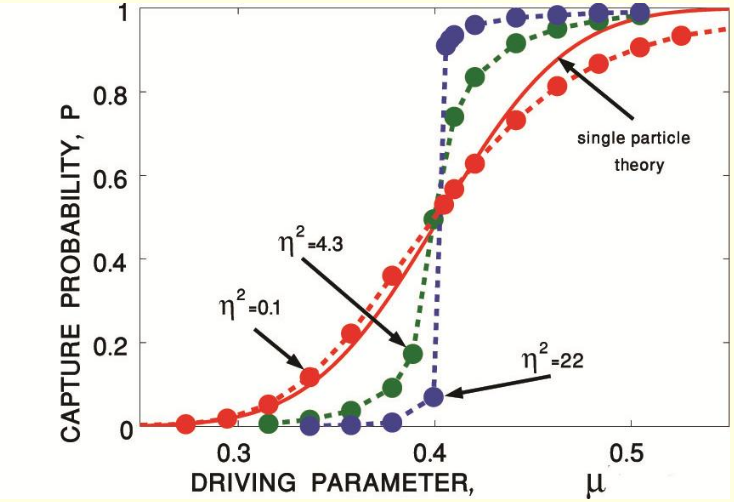
- Vlasov-Poisson Equations:

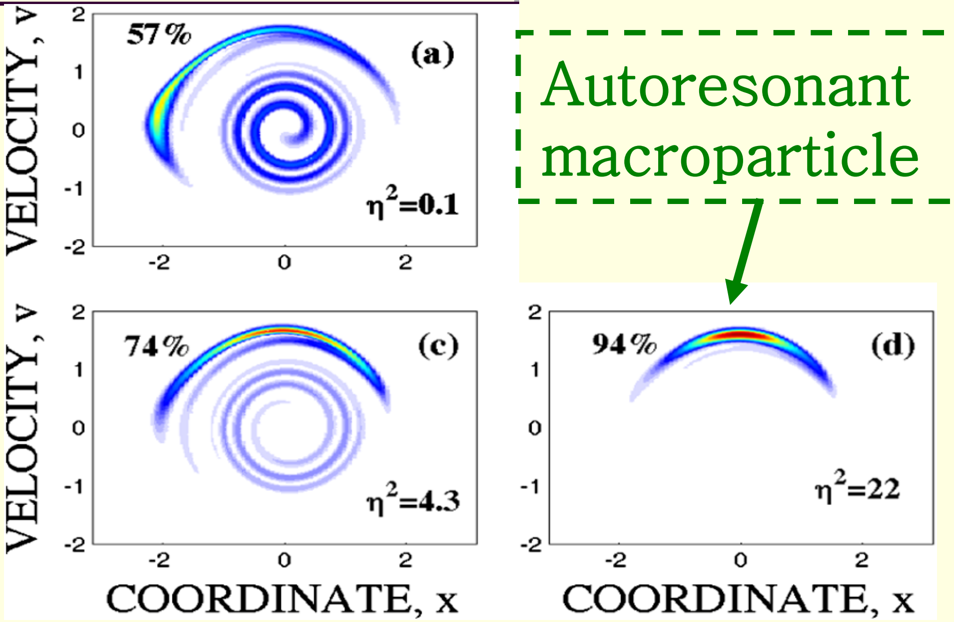
$$\begin{cases} f_t + u f_x - \left(\varphi_x^{trap} + \varphi_x^{drive} + \varphi_x^{self} \right) f_u = 0 \\ \varphi_{xx}^{self} - \chi^2 \varphi^{self} = -\eta^2 \int_{-\infty}^{\infty} f\left(u, x, t\right) du \end{cases}$$



- **Transition width decreases with density**

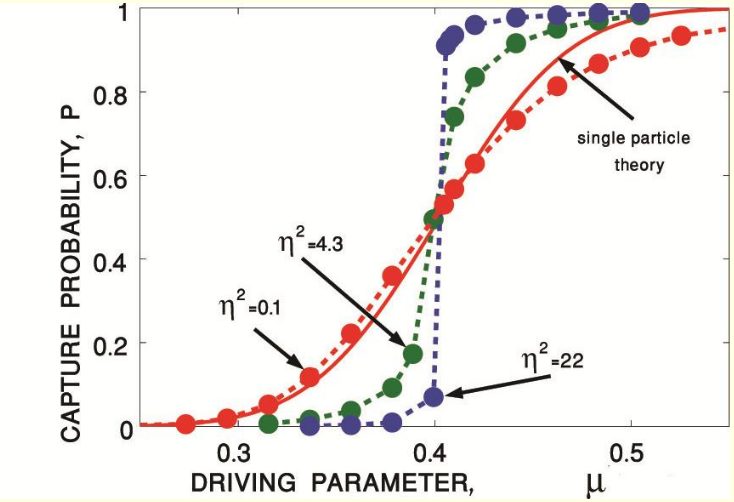
The self-field bunching effect

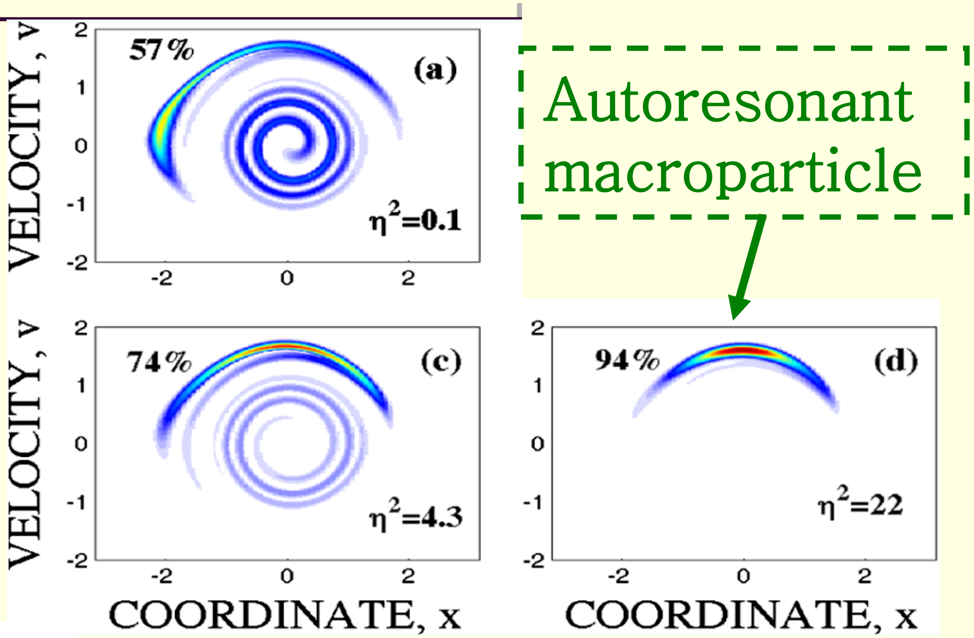




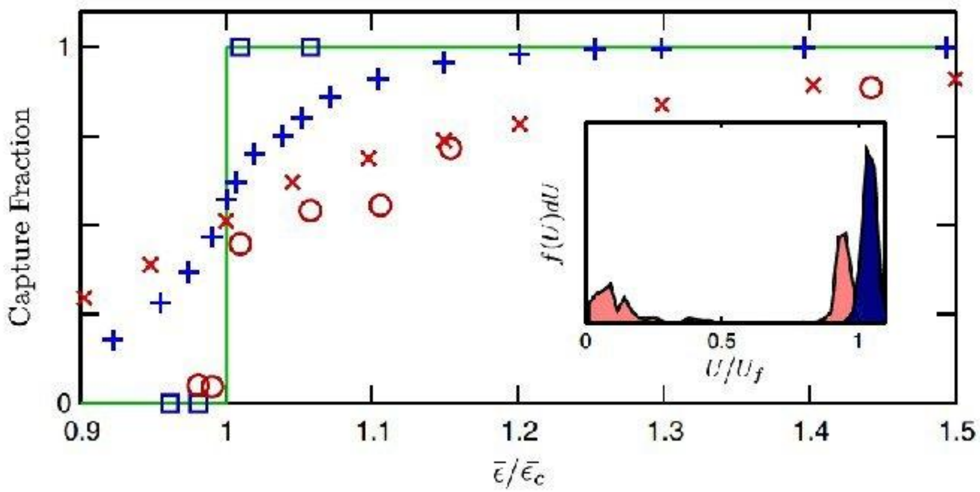
Autoresonant macroparticle

The self-field bunching effect





- Experimental results:



"To our knowledge, this is the first direct confirmation of a theory developed by Barth et al. that claims that the repulsive self-forces cause a charged plasma to stay coherent under the autoresonance drive. In conclusion … A cold, dense plasma behaves like a single-particle oscillator."

Andresen et. al. (ALPHA), PRL **106,** 025002 (2011)

The self-field effect - Theory

- Assume a continuing clustering, symmetric solution.
- Dynamics of a *test particle* in the bunch: $x_{tt} = -\sin x - \varphi_x + \varepsilon \cos \psi_d$
- Define: $\xi = x - \underbrace{x_0(t)}_{\text{centroid}}$ $\Rightarrow \begin{cases} \xi_{tt} = -\varphi_\xi - \xi \cos x_0 \\ x_{0tt} = -\sin x_0 + \varepsilon \cos \psi_d \end{cases}$
- Repulsive self-potential: $\varphi(\xi) \approx -k\,\xi^2/2$
- AR centroid solution: $x_0(t) \approx a(t) \cos \psi_d$
- Mathieu Equation: $\xi_{tt} \approx -\underbrace{\left(1 - a^2/4 - k\right)}_{\omega_{\text{eff}}^2} \xi + \frac{a^2}{4} \xi \cos 2\psi_d$

$\omega_{\text{eff}} \approx \omega_d \approx 1 \Rightarrow$ **Parametric resonance and unstable oscillations**

Parametric resonance is avoided if the self-field is strong enough

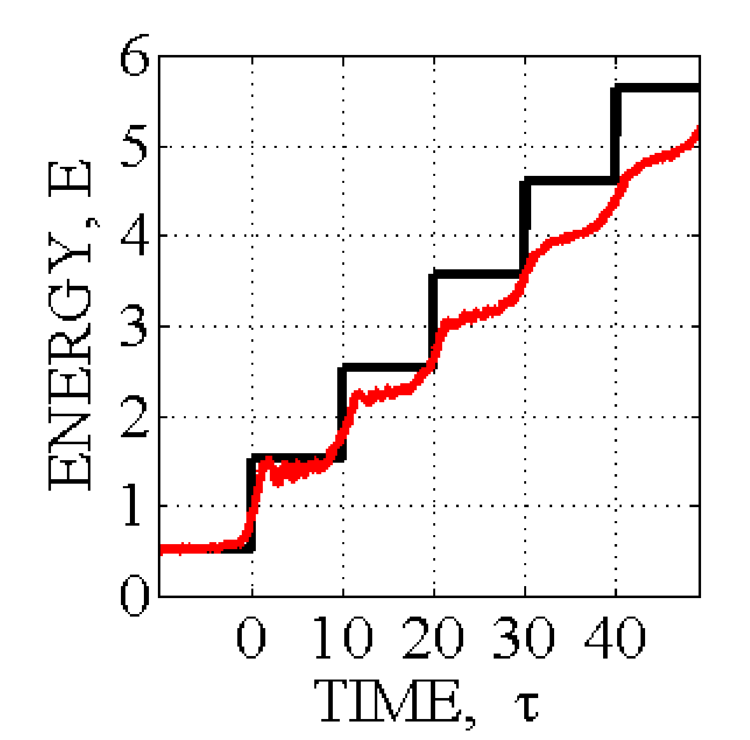
→ Autoresonant macroparticle

Barth, Friedland, Sarid, and Shagalov, PRL **103,** 155001 (2009)

D. Quantum Ladder Climbing

$$H = \frac{p^2}{2} + \frac{x^2}{2} - \beta \frac{x^4}{4} + \varepsilon x \cos\psi_d$$

$$\psi_d = t - \alpha t^2 / 2$$



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- Parameter Space

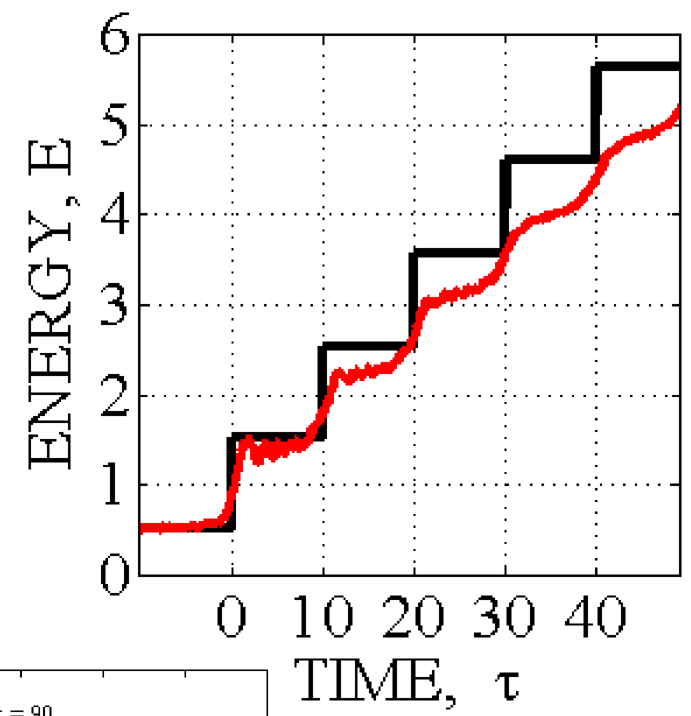
$$P_1 = \frac{\varepsilon}{\sqrt{2\alpha\hbar}} \qquad P_2 = \frac{3}{4}\frac{\hbar\beta}{\sqrt{a}}$$

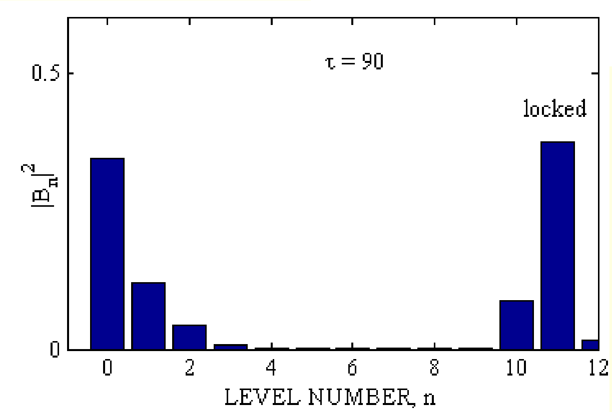
- Capture Probability:

$$P = \sum_{n=n_c}^{\infty} \left|B_b\right|^2$$

- Transition threshold

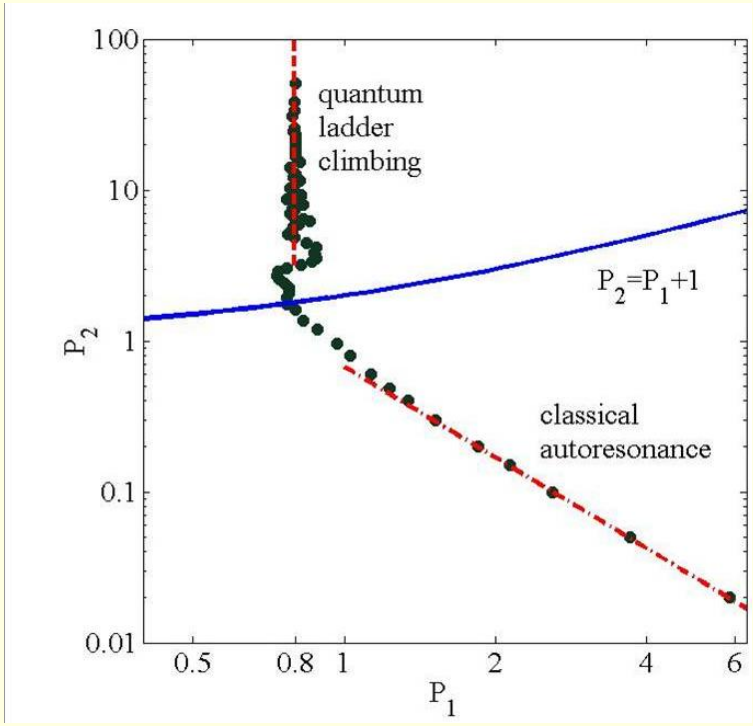
$$P_1^{cr} = P_1\left(P = 0.5\right)$$





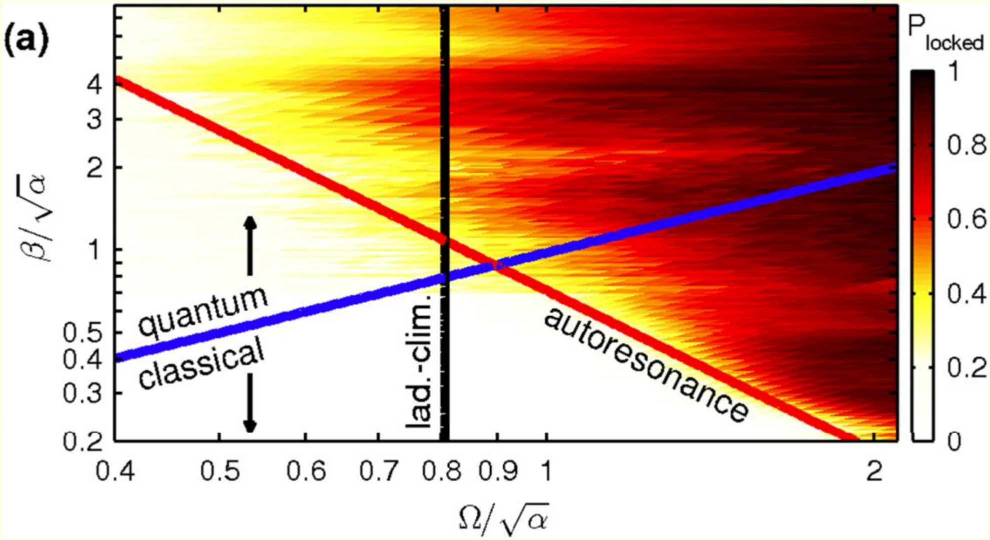
Classical to quantum transition

Theory & Simulations



Nonlinearity ⟷ Classicality

JJ Experiment



Limits:

$$\begin{cases} P_1^{cr} = 0.8 & \text{(LC)} \\ P_1^{cr} = 0.82/\sqrt{P_2} & \text{(AR)} \end{cases}$$

1. Barth, Friedland, Gat, and Shagalov, PRA **84**, 013837 (2011).
2. Shalibo, Rofe, Barth, Friedland, Bialczack, Martinis, and Katz, PRL **108**, 037701 (2012).

Summary

A. Thermal Broadening: $\text{Transition width} \propto \sqrt{T}$

B. What happens at low temperatures?
   - Quantum saturation. $T \rightarrow T_{eff}$

C. Repulsive self-field effect:
   - Transition width decreases with density!

   Autoresonant macro-particle

D. Transition to ladder climbing
   - In the quantum regime.

Thank you !

Introduction: Nonlinear Resonance

- Driven pendulum:

$$\ddot{x} + \nu \dot{x} + \sin x = \varepsilon \cos(\omega_d t)$$

- Seek steady state solutions:

$$x = b \cos(\omega_d t)$$

Autoresonace

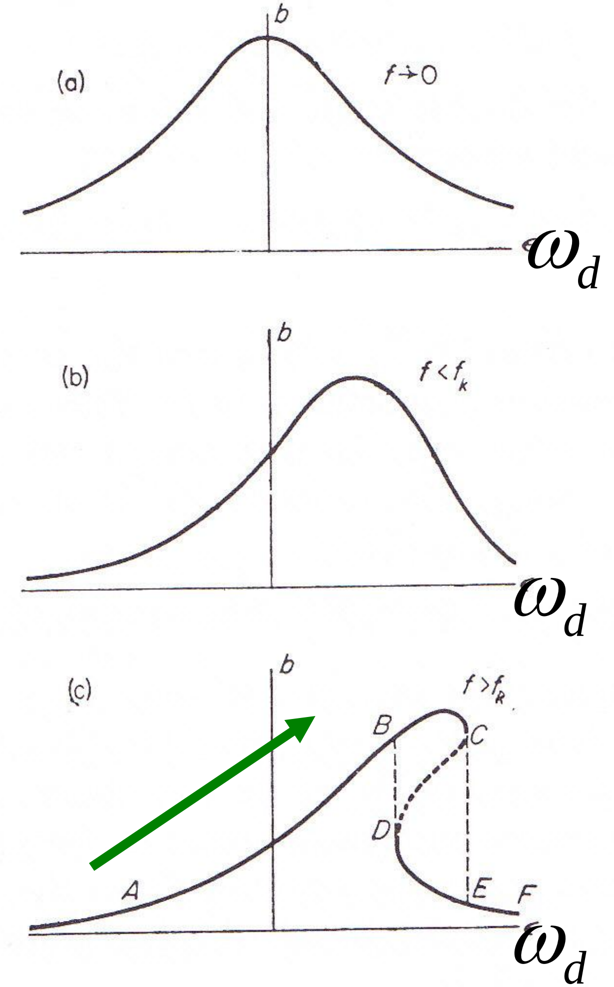


FIG. 32

L. D. Landau, Mechanics, 3rd ed. p. 88 (Oxford, 1976).

Parameter space

- Inverse Rabi frequency: $T_R = \sqrt{2m\hbar\omega_0} / \varepsilon$
- Nonlinearity time scale: $T_{NL} = (3\hbar\beta)/(4m\alpha)$
- Sweeping time: $T_S = 1/\sqrt{\alpha}$
- Driving parameter: $P_1 = \dfrac{T_S}{T_R} = \dfrac{\varepsilon}{\sqrt{2m\hbar\omega_0\alpha}}$
- Nonlinearity: $P_2 = \dfrac{T_{NL}}{T_S} = \dfrac{3\hbar\beta}{4m\sqrt{\alpha}}$

G. Marcus, L. Friedland, and A. Zigler, PRA **69**, 013407 (2004).

Quantum saturation - Theory

- Quantum Liouville equation for Wigner function in phase space

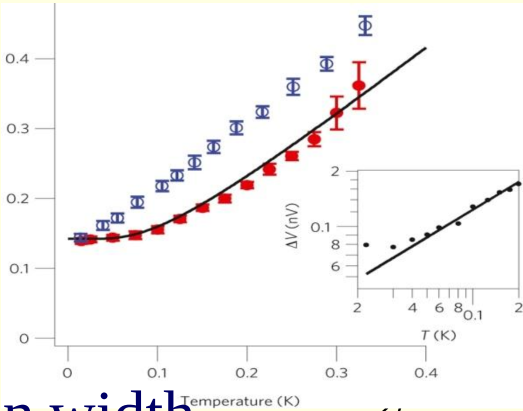
$$\underbrace{\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}-\left(\omega_0^2 x+\omega_0^2\beta x^3+\frac{\varepsilon}{m}\cos\psi_d\right)\frac{\partial f}{\partial u}}_{\text{Classical vlasov equation}}=\underbrace{-\frac{\hbar^2\omega_0^2\beta x}{4m^2}\frac{\partial^3 f}{\partial u^3}}_{\text{Quantum term}}$$

- Classical thermal distribution: $f_T(x,u,0)\propto e^{-\frac{m\omega_0^2x^2+mu^2}{2k_BT}}$
- Thermal **Wigner** distribution: $f_T(x,u,0)\propto e^{-\frac{m\omega_0^2x^2+mu^2}{2k_BT_{eff}}}$

$$T_{eff}=\frac{\hbar\omega_0}{2k_B}\coth\left(\frac{\hbar\omega_0}{2k_BT}\right)$$

$\hbar\omega_0 << k_BT$ → k_BT

$k_BT << \hbar\omega_0$ → $\frac{\hbar\omega_0}{2}$



- Minimal uncertainty → minimal transition width

Quantum Dynamics in Phase Space

- Quantum Liouville equation for Wigner function in phase space

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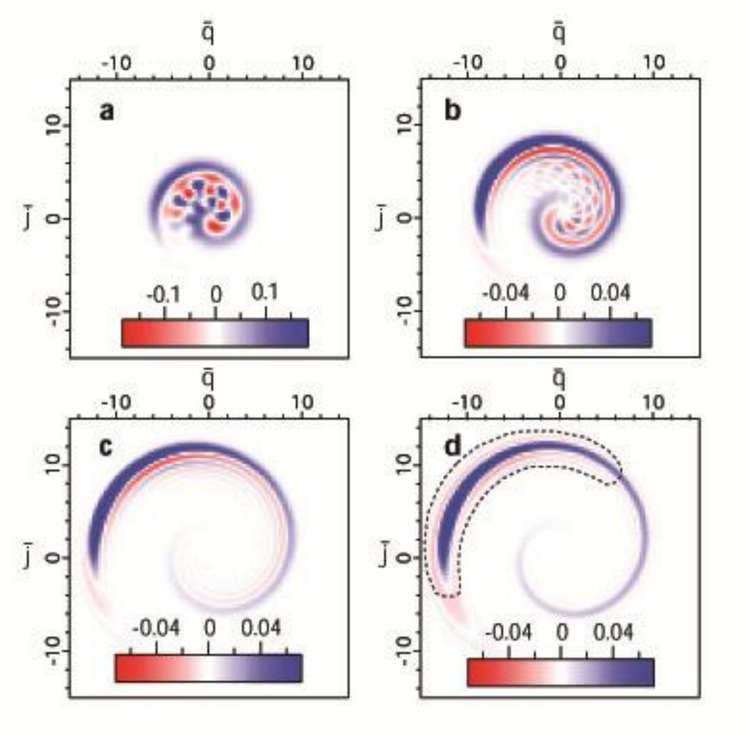
- Different nonlinearities, β:

- Quantum effects are important at sufficiently large nonlinearity.



Murch, Vijay, Barth, et. al., Nature Physics **7,** 105 (2011).

Quantum versus Classical Dynamics

- Weakly nonlinear Hamiltonian: $H = \frac{p^2}{2} + \frac{x^2}{2} - \beta \frac{x^4}{4} + \varepsilon x \cos \psi_d$

$$\psi_d = \int \omega_d dt, \quad \omega_d = 1 - \alpha t$$

- Schrodinger equation in the **Rotating Wave Approximation**

$$i \frac{dB_n}{d\tau} = \left(\frac{1}{2} P_2 n^2 - n\tau \right) B_n + \frac{P_1}{2} \left(\sqrt{n+1} B_{n+1} + \sqrt{n} B_{n-1} \right)$$

$$P_1 = \frac{\varepsilon}{\sqrt{2\alpha\hbar}} \qquad P_2 = \frac{3}{4} \frac{\hbar\beta}{\sqrt{\alpha}}$$

- Quantum Ladder Climbing vs. Classical Autoresonance (movies)

Rotating frame Wigner function

- Quantum Liouville equation: $$\frac{\partial f}{\partial \tau}+\frac{\partial G}{\partial P}\frac{\partial f}{\partial Q}-\frac{\partial G}{\partial Q}\frac{\partial f}{\partial P}=\frac{\lambda^2}{4}\hat{D}f$$
- Hamiltonian: $$G=\frac{\tau}{2}(Q^2+P^2)-\frac{1}{4}(Q^2+P^2)^2+\mu Q$$

$$\hat{D}=\left(Q\frac{\partial}{\partial P}-P\frac{\partial}{\partial Q}\right)\left(\frac{\partial^2}{\partial Q^2}+\frac{\partial^2}{\partial P^2}\right),\qquad \mu=\frac{1}{2}P_1P_2^{1/2}$$

- Effective Plank constant: $[Q,P]=i\lambda,\qquad \lambda=\frac{1}{2}P_2=\frac{3\hbar\beta}{8m\sqrt{\alpha}}$
- Classical limit: $\lambda\to 0\qquad (P_2\ll 1)$
- Thermal **Wigner** distribution: $$f_0(Q,P)=\frac{1}{2\pi\sigma^2}e^{-\frac{Q^2+P^2}{2\sigma^2}}$$
- Effective temperature: $$\sigma^2=\lambda\frac{k_BT_{\mathrm{eff}}}{\hbar\omega_0}=\frac{\lambda}{2}\coth\left(\frac{\hbar\omega_0}{2k_BT_{\mathrm{eff}}}\right)$$